

Entanglement Entropy in Extended Quantum Systems *

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After a brief introduction to the concept of entanglement in quantum systems, I apply these ideas to many-body systems and show that the von Neumann entropy is an effective way of characterising the entanglement between the degrees of freedom in different regions of space. Close to a quantum phase transition it has universal features which serve as a diagnostic of such phenomena. In the second part I consider the unitary time evolution of such systems following a ‘quantum quench’ in which a parameter in the hamiltonian is suddenly changed, and argue that finite regions should effectively thermalise at late times, after interesting transient effects.

I. INTRODUCTION

Entanglement is one of the most fundamental features of quantum mechanics, yet in some ways the most mysterious. In this talk I will discuss some of the recent developments in understanding its role in many-body systems extended in space, focussing on two main topics: the universal properties of entanglement entropy near quantum critical points, and the behaviour of entanglement, and, more generally, correlation functions, after what is termed a ‘quantum quench’.

Most of the work I will be presenting was carried out in collaboration with P. Calabrese [1, 2, 3] and others. For a more general perspective the reader is referred to a recent review by Amico *et al.* [4].

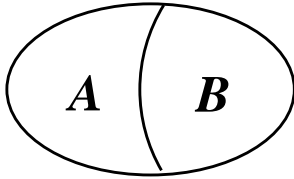


FIG. 1: Schematic depiction of the decomposition of the Hilbert space. Later these will also represent regions of d -dimensional space.

II. ENTANGLEMENT ENTROPY NEAR QUANTUM CRITICAL POINTS

A. Bipartite quantum entanglement

Consider a general quantum system prepared in a pure state $|\Psi\rangle$, so that it has density matrix $\rho = |\Psi\rangle\langle\Psi|$. We suppose that the Hilbert space can be written as a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ (see Fig. 1). We imagine two observers (traditionally named Alice and Bob) such that Alice can make observations only in \mathcal{H}_A (that is corresponding to linear operators of the form $\mathcal{O}_A \otimes 1_B$), and correspondingly for Bob. In general Alice’s observations are entangled with those of Bob.

One of the most useful mathematical results in understanding how to quantify entanglement is that of Schmidt decomposition, which is based on the property of singular value decomposition for matrices. It states that any pure state $|\Psi\rangle$ may be written as

$$|\Psi\rangle = \sum_j c_j |\psi_j\rangle_A \otimes |\psi_j\rangle_B,$$

where $|\psi_j\rangle_{A,B}$ are orthonormal states in \mathcal{H}_A and \mathcal{H}_B respectively, and $\sum_j |c_j|^2 = 1$. Moreover the c_j can be chosen to be real and ≥ 0 . (Note that there is only one sum here: for each state in \mathcal{H}_A there is just one state in \mathcal{H}_B .)

One measure of the entanglement in $|\Psi\rangle$ is the *entropy*

$$S \equiv - \sum_j c_j^2 \log c_j^2.$$

If $c_1 = 1$ and all the rest vanish, $|\Psi\rangle$ is a product state and is unentangled (although there may still be classical correlations.) If, on the other hand, all the c_j are equal, then S takes its maximal value, given by the logarithm of the smaller of the dimensions of \mathcal{H}_A and \mathcal{H}_B . For example, if each space is a direct product of N qubits (spin $\frac{1}{2}$ degrees of freedom) then the maximal entanglement entropy is $N \log 2$.

Equivalently, we can define the entanglement entropy as the von Neumann entropy $S_A = -\text{Tr}_A \rho_A \log \rho_A$ of Alice’s density matrix $\rho_A = \text{Tr}_B \rho$. Evidently $S_A = S_B = S$.

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Other measures of entanglement exist [4], but the entropy has several nice properties: additivity, convexity, basis independence. In quantum information theory [5], it gives the maximum efficiency of conversion of partially entangled to maximally entangled states that Alice can achieve using only local operations in her part of the Hilbert space. From a computational point of view, it gives the amount of classical information required to specify the reduced density matrix ρ_A . This important, for example, in the density matrix renormalisation group (DMRG).

However, in this talk I want to use the entanglement entropy as a basis-independent means of characterising quantum phase transitions. This is particularly important in those cases where it is not clear what the order parameter is, or which correlation functions should become long-ranged at the transition. (Indeed, for topological transitions there may be no identifiable order parameter.) I shall consider the case when the degrees of freedom of the quantum system are distributed over some large region \mathcal{R} in d -dimensional euclidean space, and the hamiltonian H contains only short-range interactions, for example a quantum spin system, or, more generally a UV-cutoff quantum field theory. (I shall focus solely on the universal properties near the phase transition: these, as K. Wilson and others taught us, are all encoded in the field theory description.) The subspace \mathcal{H}_A will consist of the degrees of freedom in some large (compact) subset A of \mathcal{R} , and we shall assume that the whole system is in a pure state, usually the ground state $|0\rangle$ of H . However, it will be useful also to consider the case when the whole system is in a thermal mixed state with $\rho \propto e^{-\beta H}$.

The main question I will address is how the entanglement entropy depends on the size and geometry of the region A , and on the universality class of the critical behaviour.

B. Entanglement entropy from the path integral

“Quantum mechanics is just statistical mechanics in one more dimension” (M.E. Fisher)

From an analytic perspective, the Schmidt decomposition is very difficult to carry out except in simple cases, and we employ a different route via the path integral. As Feynman taught us, the thermal density matrix $\propto e^{-\beta H}$ may be written as a path integral in an imaginary time interval $(0, \beta)$ (throughout we adopt units with $\hbar = 1$.) This is illustrated in Fig. 2. The density matrix is correctly normalised by the partition function Z as shown in Fig. 3. However, we need the reduced density matrix ρ_A . This is found by sewing together only those parts of the upper and lower edges corresponding to the region B (Fig. 4). In order to compute the entropy $S_A = -\text{Tr}_A \rho_A \log \rho_A$ we use a device reminiscent of the ‘replica trick’ in disordered systems: we compute $\text{Tr} \rho_A^n$ for positive integral

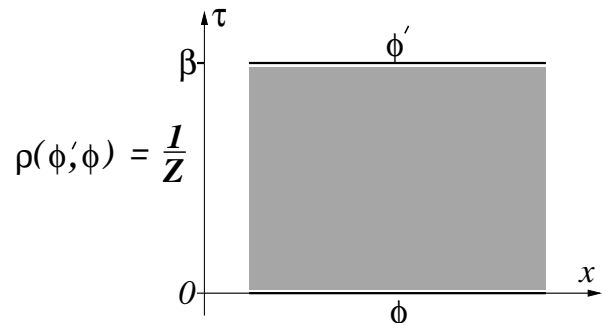


FIG. 2: The path integral in space x and imaginary time τ . The rows and columns of the density matrix are labelled by the values of the fields (degrees of freedom) at $\tau = 0$ and β respectively.



FIG. 3: The partition function Z is obtained by sewing together the top and bottom edges of the world sheet in Fig. 2, that is by identifying the fields and integrating them.

n , analytically continue in n , and evaluate

$$S_A = -\text{Tr} \rho_A \log \rho_A = - \left. \frac{\partial}{\partial n} \right|_{n=1} \text{Tr} \rho_A^n.$$

For positive integer n , this is given by taking n copies of Fig. 4 and sewing the edges along A in a cyclic fashion, as shown in Fig. 5. For $d = 1$ (which is the case we shall consider here) this expresses $\text{Tr} \rho_A^n$ as a path integral on an n -sheeted Riemann surface, with branch points at the ends of the interval A (if A has several disjoint parts, there are several branch points.)

1. High temperature limit

Before proceeding further, it worth understanding how this formalism gives the expected answer at high temperatures. Consider the case where A is an interval of length ℓ in a system of total length L , with $\ell \ll L$. Further



FIG. 4: The reduced density matrix is obtained by sewing together the top and bottom edges of the world sheet in Fig. 2, along only the parts corresponding to region B .

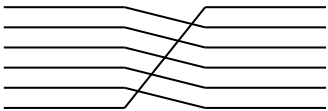


FIG. 5: A view along the edges A of n copies of Fig. 4, showing how they are to be sewn together to give $\text{Tr} \rho_A^n$.

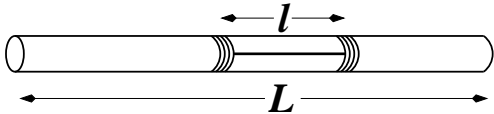


FIG. 6: The world sheet for the path integral for $\text{Tr} \rho_A^n$ in the high temperature limit. In the central section, corresponding to the region A, it winds around n times.

suppose that $\beta \ll \ell$. The path integral is then over a very narrow cylinder (compared to its length) as shown in Fig. 6. In this limit we expect the result of the path integral to factorise:

$$Z_n \approx Z_1(\ell, n\beta) Z_1(L - \ell, \beta)^n,$$

where $Z_1(L, \beta)$ is the partition function for a system of length L at inverse temperature β . Thus

$$\text{Tr} \rho_A^n \sim \frac{Z_1(\ell, \beta)}{Z_1(\ell, \beta)^n} \sim \frac{\exp -n\beta F_A(n\beta)}{\exp(-n\beta F_A(\beta))}$$

where $F_A(\beta)$ is the usual Helmholtz free energy of region A. Differentiating with respect to n at $n = 1$, we then find

$$S_A \sim \beta(E_A - F_A).$$

This shows that in this limit, the von Neumann entropy becomes the usual thermodynamic entropy, as expected.

C. The critical case in $d = 1$

Suppose the 1d system is at a quantum critical point with dynamic exponent $z = 1$ (that is, linear dispersion relation $\omega = v|k|$ at low energies). Then dimensional analysis (with $\hbar = v = 1$) implies that $F_A(\beta) \sim -\pi c \ell / 6\beta^2$. This is just the one-dimensional version of Stefan's law: the constant $c = 1$ for a single species of boson. We conclude that for $\ell \gg \beta$

$$\text{Tr} \rho_A^n \sim \exp \left[-\frac{\pi c}{6\beta} \left(n - \frac{1}{n} \right) \ell \right].$$

The above considerations do not appear to shed much light on the case we wish to consider, namely zero temperature. However, in this case the limit $\ell \gg \beta$ is related to the opposite one $\beta \rightarrow \infty$ by *conformal symmetry*: the mapping $z \rightarrow (\beta/2\pi) \log z$ converts exponential decay along the cylinder into power law decay at $T = 0$. Thus, at zero temperature we have

$$\text{Tr} \rho_A^n \sim \ell^{-(c/6)(n-1/n)}.$$

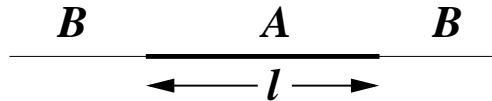


FIG. 7: An interval A of length ℓ . When the correlation length $\xi \ll \ell$, the entanglement arises from regions of length $O(\xi)$ near the boundaries between A and B.

Taking the derivative we see that

$$S_A \sim (c/3) \log \ell. \quad (1)$$

For a general conformal field theory (CFT), c is the central charge. (1) was first found in 1994 by Holzhey *et al.* [6], who called it the geometric entropy.

Note that this implies that, even at a quantum critical point, the entropy grows only logarithmically in the length ℓ , as opposed to its maximum allowed behaviour which is $O(\ell)$. This accounts for the success of the DMRG method in $d = 1$. This logarithmic growth is apparently not restricted to critical points with $z = 1$: for example it also holds in random spin chains [7].

(1) is just one example of a plethora of universal results which can be found using CFT methods. For example, one can explore the cross-over between finite and zero temperature when $\ell \sim \beta$ to find the elegant formula [1, 8]

$$S_A \sim (c/3) \log \left((\beta/\pi) \sinh(\pi\ell/\beta) \right).$$

D. Finite correlation length in $d = 1$

The above CFT considerations apply only at the quantum critical point. However, the behaviour of the entanglement entropy close to such a point may be deduced from this by scaling arguments. For example, consider the case when A is an interval of length ℓ in an infinite system (Fig. 7). When the correlation length ξ is finite and $\ll \ell$, we expect that the entanglement arises from regions of length $O(\xi)$ near the boundaries between A and B. Since the scaling at the critical point is logarithmic we may therefore conjecture that, in this limit,

$$S_A \sim 2 \times (c/6) \log \xi, \quad (2)$$

and that in general there is a contribution $(c/6) \log \xi$ from each contact point between A and B. For the case of a single such point, when A and B are both semi-infinite lines, this has been derived exactly using the corner transfer matrix [1, 9], for a large class of integrable models. Recently the leading corrections to (2) have been computed [10]. These are of the form $O(e^{-2\ell/\xi})$ and they appear to be rather universal.

E. Higher dimensions $d > 1$

In higher dimensions, the natural conjecture for the entanglement entropy (given that S_A should equal S_B)

is that it is proportional to the ‘area’ $|\partial A|$ of the boundary between the two regions. This has been verified in various models (although logarithmic factors are also possible) and even proved rigorously [11] starting from reasonable assumptions about the behaviour of correlations. On dimensional grounds the coefficient of the area law should go as a^{1-d} where a is the short-distance cutoff, and therefore be non-universal. However, renormalisation group arguments similar to those used to analyse the scaling behaviour of the free energy imply that there should be a universal term proportional to $\xi^{1-d}|\partial A|$ hidden behind this [1].

It is interesting to note that these area-dependent terms may cancel in more complicated entanglement combinations. For example, if A and B are different subregions of \mathcal{R} , then one may conjecture that

$$S_{A \cup B} + S_{A \cap B} - S_A - S_B$$

is universal, and, at the critical point, depends only on the geometry and some universal constants of the critical theory.

III. TIME-DEPENDENCE AFTER A QUANTUM QUENCH

In the second part of this talk I want to discuss a subject which is intimately related to entanglement properties of quantum many-body systems, but is somewhat more general. Let us suppose that we prepare a system at time $t = 0$ in a pure state $|\Psi_0\rangle$, which we usually take to be the ground state of some translationally invariant hamiltonian H_0 with a gap m_0 to the first excited state. For times $t > 0$ we then evolve the state according to a *different* hamiltonian H (which doesn’t commute with H_0). Note that this evolution is *unitary*: no dissipation or noise. This protocol has been termed a ‘quantum quench’.

We may then ask how the reduced density matrix ρ_A of some finite part of the system, its entropy S_A and correlation functions of local operators $\mathcal{O}(x)$ with $x \in A$ evolve. In particular, does ρ_A reach a stationary state, and if so how is it characterised?

For traditional solid state systems the assumption of unitary evolution usually breaks down so rapidly that these kind of questions have hardly been addressed in the past, except in a few cases of integrable spin chains [12]. However recent experiments on cold atoms in optical lattices have shown that it is possible to maintain coherence over measurable time intervals. This has prompted more recent work, both in integrable systems and more generally [2, 3, 13, 14]. These detailed considerations all lead to the prediction of the following rather simple physical effects.

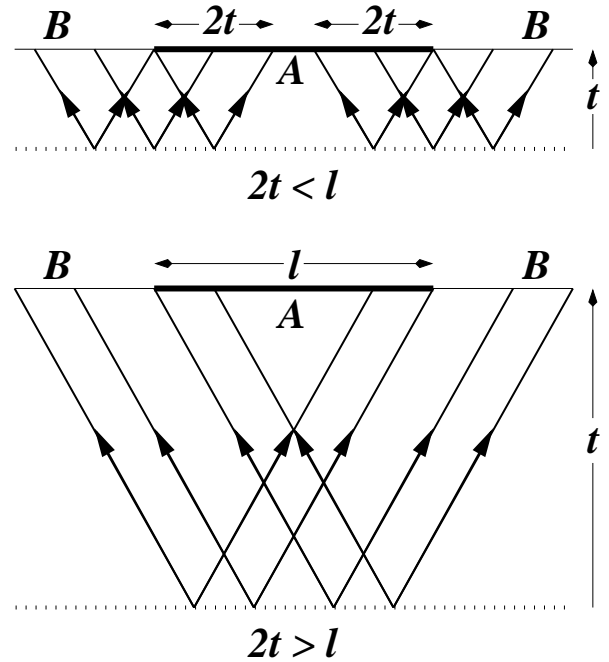


FIG. 8: Quasiparticles are emitted at $t = 0$, and subsequently move classically. Left and right-moving pairs of particles are entangled, and if they arrive on A and B respectively they cause entanglement of these two regions.

A. The horizon effect

Since $|\Psi_0\rangle$ has (extensively) higher energy than the ground state of H , it acts as a source of quasiparticles of H . The analysis of simple models shows that subsequently they move classically. This is illustrated in Fig. 8, for the simplest case in $d = 1$ when all particles travel at the same speed v . Entanglement between regions A and B (which is very small initially) arises when two entangled particles emitted from nearby points arrive on A and B respectively. From the upper diagram in Fig. 8, corresponding to early times $t < \ell/2v$, we see that the number of such pairs of particles, and therefore the degree of entanglement, increases linearly with t . On the other hand for $t > \ell/2v$ is saturates at a value proportional to ℓ . This is what is found in explicit calculations. The coefficient of the linear term depends on the initial state.

1. Correlation functions

In general it is found that one-point functions $\langle \mathcal{O}(x, t) \rangle$ of local operators (for example spins in a quantum spin chain where $|\Psi_0\rangle$ breaks the spin-reversal symmetry but H and its ground state do not), decay exponentially fast towards their values in the ground state of H . The physical reason for this is illustrated in Fig. 9. On the other hand the behaviour of a two-point function $\langle \mathcal{O}(x_1, t) \mathcal{O}(x_2, t) \rangle$ is governed by the horizon effect. Up to

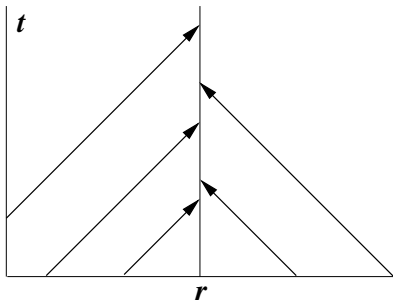


FIG. 9: A local observable, for example a quantum spin, is subject to incoherent radiation of quasiparticles emitted from different spatial points. Its value therefore flips according to a Poisson process, and its expectation value relaxes exponentially.

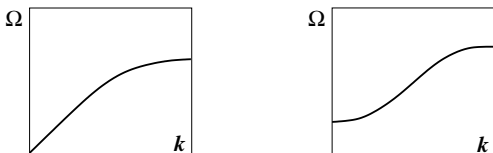


FIG. 10: Two examples of dispersion relations. In each case, the quasiparticles move with their group velocity given by the slope of this curve. The fastest moving particles are responsible for the horizon effect, and the slowest particles for the approach to the limiting behaviour at late times.

time $t \sim |x_2 - x_1|/2v$ its connected part does not change from its initial form which is very short-ranged. For wider separations $\langle \mathcal{O}(x_1, t) \mathcal{O}(x_2, t) \rangle \sim \langle \mathcal{O}(t) \rangle^2$, which decays exponentially in time. At the time $t \sim |x_2 - x_1|/2v$ the two points fall inside the horizon and the connected correlation function becomes non-zero. In the case when all quasiparticles have the same speed, the full correlation function then becomes time-independent, and, since it was previously decaying exponentially with t , now exhibits exponential decay in the separation $|x_2 - x_1|$.

2. General dispersion relation

These physical considerations allow us to understand the behaviour in more realistic situations where lattice effects or a gap modify the dispersion relation. Examples are shown in Fig. 10. The fact that the quasiparticles, once emitted, travel classically means that it is their group velocity which is important. Since this can vanish, for example at the zone boundary, this is responsible for a very slow (power law + oscillations) approach to the asymptotic limit at large t , which is actually seen in exact calculations and numerical work.

B. Thermalisation at late times

We have argued that calculations in various solvable models suggest that correlation functions for $x \in A$, and by implication the reduced density matrix ρ_A , become stationary at late times. But what is their form? In these cases it turns out that, at least in the case where the gap m_0 in the spectrum of H_0 is large, ρ_A has a *thermal* form $\propto e^{-\beta_{\text{eff}} H}$, where β_{eff}^{-1} is an effective temperature dependent on m_0 among other things.

In order to understand this, first consider a very simple example: a simple harmonic oscillator quenched from frequency ω_0 (and initially in the ground state), to frequency ω . It is a simple calculation to evaluate the overlap between the ground state $|\Psi_0\rangle$ and a typical eigenstate of H with energy E . For $\omega_0 \gg \omega$ we find

$$\langle \Psi_0 | E \rangle \propto \exp(-\beta_{\text{eff}} E/2),$$

where $\beta_{\text{eff}} \sim 4/\omega_0$, so the matrix elements of the time-dependent density matrix have the form

$$\langle E | \rho(t) | E' \rangle \sim e^{-\beta_{\text{eff}}(E+E')/2} e^{i(E-E')t}.$$

Note that since the energy differences are all multiples of ω , $\rho(t)$ does not tend towards a stationary value – it oscillates as expected!

However in an extensive system, which we can consider as a set of oscillators, one for each quasiparticle mode k , if we consider a finite region A of size $\ll t/v$ as $t \rightarrow \infty$, we need to integrate over all the k -modes. If the dispersion relation has the form $\omega_k = m + O(k^2)$ the different modes destructively interfere except at $k = 0$. The result is that correlation functions, and $\rho_A(t)$, become stationary as $t \rightarrow \infty$ as if they were at finite temperature. In fact a more careful calculation, valid not just in the large m_0 limit, gives

$$\beta_{\text{eff}} = (4/m) \tanh^{-1}(m/m_0).$$

Similarly, we find that the extensive part of S_A saturates at a value equal to the thermodynamic entropy at this effective temperature.

Although the above simple argument relied on a non-zero gap $m > 0$, in fact (as long as interactions are present) the above result appears to hold (for example in a CFT in 1+1 dimensions) even in the gapless case.

It should be stressed that this effective thermalisation occurs despite the fact that the system as a whole remains in a pure state, and there is no ergodicity or coupling to a heat bath – the effect arises solely as a consequence of quantum interference and entanglement.

Summary

In this talk I have argued (a) that entanglement entropy provides a useful order-parameter independent diagnostic of quantum phase transitions, with many universal features, and (b) that after a quantum quench,

there are interesting transient phenomena like the horizon effect, and that at late times finite regions should behave as though they are in thermal equilibrium.

There are many open questions, particularly in regard to part (b), and whether it holds only for theories which admit a quasiparticle picture. In addition, these kinds of calculations need to be repeated for more realistic mod-

els, and also for quenches through the critical point into an ordered phase, where the relation to later-time coarsening effects needs to be understood.

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